THE PROBLEM OF PLANE UNSTEADY MOTION OF GROUND WATERS

(K ZADACHE O PLOSKOM NEUSTANOVIVSHEMSIA DVIZHENII GRUNTOVYKH VOD)

PMM Vol.23, No.5, 1959, pp. 954-957

V. G. PRIAZHINSKAIA
(Tomsk)

(Received 16 March 1959)

We consider the plane problem of the motion of a liquid in a porous medium under the action of gravity forces, when the liquid occupies a certain semi-infinite region $G(t)$, bounded by a curve $\Gamma(t)$, without multiple points, extending to infinity in both directions. With the passage of time the region $G(t)$ will change; the shape of the region $G(0)$ at the initial instant of time is assumed known. Moreover, it is assumed that the boundary $\Gamma(0)$ of the region has the x-axis as an asymptote; it is natural to suppose that, in the given conditions, this property is retained at any instant of time t during the flow of the liquid under the action of the forces of gravity.

It is assumed that the pressure p on the contour $\Gamma(t)$ is constant at all stages of the motion and is equal to zero; we shall assume that the motion of the liquid obeys Darcy's law.

Galin [1] reduced this problem to a nonlinear boundary problem in the theory of analytic functions, to the solution of which the author applied the method of successive approximations.

We give below a derivation of the boundary condition which is different from that in paper [1]; the problem is reduced to a certain integro-differential equation, and thence to a system of nonlinear integral equations. Under specified conditions the existence of a solution of this system is established by a method analogous to that applied to another problem in a paper [2] by Kufarev and Vinogradov.

Suppose that the function $z = z(w, t)$, which we normalize by the condition

$$\lim [z(w, t) - w] = 0 \qquad \text{when } w \to \infty, \qquad \operatorname{Im} w \leqslant 0$$

conformally maps the lower half-plane $\operatorname{Im} w \leqslant 0$ on to the region $G(t)$. We shall assume that $z(w, t)$ can be represented in the form

$$z(w, t) = w + \Psi(w, t)$$

where $\Psi(w, t)$ is a function which is holomorphic in the lower half-plane $\operatorname{Im} w \leqslant 0$ and, for sufficiently large values of $|w|$, $\operatorname{Im} w \leqslant 0$, satisfies the conditions

$$|\Psi(w, t)| \leqslant \frac{1}{|w|^{\mu}}, \qquad |\Psi_t'| \leqslant \frac{1}{|w|^{\mu}}, \qquad |\Psi_w'| \leqslant \frac{1}{|w|^{\mu}} \qquad (0 < \mu < 1) \tag{1}$$

We shall assume that the function $z_0(w) = z(w, 0)$ is given, in so far as $G(0)$ is known.

According to Darcy's law, the pressure is related to the complex potential of the motion $X^*(z, t)$ by the following equation:

$$X^*(z, t) = -\frac{k}{\rho g} X_1^*(z, t) + ikz \qquad (X_1^* = p + iq)$$

(see [3], Chapter XV), where k is the filtration coefficient, ρ - the density of the fluid, g - the acceleration due to gravity, p - the pressure in the fluid and q - the function which is the harmonic conjugate of p. According to the conditions of the problem

$$p = \operatorname{Re} X_1^*(z, t) = 0 \quad \text{on} \quad \Gamma(t)$$

Introducing the notation

$$X^*(z(w, t), t) = X(w, t), \qquad X_1^*(z(w, t), t) = X_1(x, t)$$

let us express the complex potential of the motion in terms of the variable w in the form [1]

$$X(w, t) = -kiw + kiz(w, t) \tag{2}$$

Turning to the derivation of the boundary conditions for the function $z(w, t)$, let us compare the two expressions for the velocity of the fluid particle at the point z(cf. [2]):

$$v = \frac{1}{m}\frac{\overline{\partial X^*}}{\partial z} = \frac{ik}{m}\frac{\overline{\partial w}}{\partial z} - \frac{ik}{m}$$

where m is the porosity of the soil.

On the other hand, the velocity of the fluid particle is expressible in the form

$$v = \frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial w}\frac{dw}{dt}\,.$$

Equating the expressions so obtained for the velocity, after multiplying by $\partial z/\partial w$, we find that

$$\frac{ik}{m} - \frac{ik}{m}\overline{\frac{\partial z}{\partial w}} = \frac{\partial z}{\partial t}\overline{\frac{\partial z}{\partial w}} + \left|\frac{\partial z}{\partial w}\right|^2 \frac{dw}{dt}$$

Hence, after multiplying by i, we obtain

$$\operatorname{Re}\left[\frac{k}{m}\frac{\partial z}{\partial w}+i\frac{\overline{\partial z}}{\partial t}\frac{\partial z}{\partial w}\right]=\frac{k}{m}$$

or

$$\operatorname{Re}\left[i\frac{\partial Z}{\partial w}\frac{\overline{\partial Z}}{\partial t}\right]=\pm 1 \text{ when } \eta=0 \qquad \left(Z=z(w,\ t)+kit\ ,\ w=\xi+i\eta\right) \tag{3}$$

Dividing both sides of equation (3) by $|Z_w|^2$, we have

$$\operatorname{Re}\left[i\frac{Z_t}{Z_w}\right]=-|\nu|^2 \qquad \left(\nu=\frac{1}{Z_w}\right) \tag{4}$$

In what follows it will be assumed that the function $\nu(w,\ t)$ satisfies the Holder condition on a real straight line (see [4]).

According to the boundary condition (4) the function iZ_t/Z_w is analytic in the half-plane $\operatorname{Im} w < 0$ and, allowing for condition (1), can be represented as an integral by means of the Schwarz formula for the half-plane (see [3], Chapter VI)

$$i\frac{Z_t}{Z_w}=P, \qquad P(w,\ t)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}|\nu(w,\ t)|^2\frac{d\omega}{\omega-w} \tag{5}$$

Here the integral is to be regarded in the sense of its principal value. Writing (5) in the form

$$Z_t+iZ_wP=0 \tag{6}$$

and differentiating with respect to w, we obtain an integro-differential equation for $\nu(w,\ t)$:

$$\frac{\partial\nu}{\partial t}=i\left[\nu\frac{\partial P}{\partial w}-P\frac{\partial\nu}{\partial w}\right] \tag{7}$$

where P is expressed in terms of the boundary values of the function $\nu(w,\ t)$ by formula (5).

If $\nu(w,\ t)$ is the solution of equation (7) which satisfies the initial condition $\nu(w,\ 0)=\nu_0(w)$, then the required function $Z(w,\ t)$ is determined by the formula

$$Z(w,\ t)=\int_0^w\frac{d\omega}{\nu(\omega,t)}+K(t) \qquad \left(K(t)=-i\int_0^t\frac{P(0,\ t)}{\nu(0,\ t)}dt+Z_0(0)\right) \tag{8}$$

Making the substitution

$$\nu(w,\ t)=1+u(w,\ t) \qquad \left(|u(w,\ t)|\leqslant\frac{1}{|w|^{\mu}}\ ,\quad 0<\mu<1\right) \tag{9}$$

(for sufficiently large values of $|w|$) in equation (7), we obtain an equation for $u(w,\ t)$:

$$\frac{\partial u}{\partial t}=i\left[u\frac{\partial Q}{\partial w}-Q\frac{\partial u}{\partial w}+2\frac{\partial u}{\partial w}+\frac{\partial Q}{\partial w}\right], \qquad \left(Q(w,t)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}|u(\omega,t)|^2\frac{d\omega}{\omega-w}\right) \tag{10}$$

The solution of equation (10) will be sought in the form of an integral of Cauchy type along the line C_w (Im $w = \delta$, $\delta > 0$):

$$u(w,t)=\int_{C_\xi}\frac{X(\xi,t)\,d\xi}{w-x(\xi,t)} \tag{11}$$

The functions $x(\xi, t)$, $X(\xi, t)$ in this expression are assumed to be defined in the region D_{t_0} ($|t| \leqslant t_0$, $t_0 > 0$, ξ C_ξ) and satisfy the following conditions.

1. The function $x(\xi, t) - \xi$ and its derivatives up to the second order inclusive, are holomorphic with respect to t and uniformly bounded:

$$|x(\xi,t)-\xi|\leqslant C, \qquad \left|\frac{\partial^{k+l}x(\xi,t)}{\partial\xi^k\partial t^l}\right|\leqslant B \qquad (k,l=0,1,2,\ 1\leqslant k+l\leqslant 2) \tag{12}$$

For sufficiently large values of $|\xi|$

$$|x(\xi,t)-\xi|\leqslant\frac{C}{|\xi|} \tag{13}$$

2. The function $X(\xi, t)$ is holomorphic relative to t and satisfies the conditions:

$$|X(\xi_1,t)-X(\xi_2,t)|\leqslant H|\xi_1-\xi_2|^\mu, \quad \left|\frac{\partial^k X(\xi,t)}{\partial t^k}\right|\leqslant H \qquad (k=0,1,2) \tag{14}$$

and, for sufficiently large $|\xi|$, the condition

$$\left|\frac{\partial^k X}{\partial t^k}\right|\leqslant\frac{H}{|\xi|^\mu} \tag{15}$$

Making use of expression (11), we find from (10) that

$$Q(w,t)=\int_{C_\zeta}\int_{C_\eta}\frac{2X\overline{Y}d\eta d\zeta}{(x-\overline{y})(x-w)} \qquad \begin{pmatrix} X=X(\xi,t), & x=x(\xi,t) \\ Y=X(\eta,t), & y=x(\eta,t)\end{pmatrix} \tag{16}$$

Equation (10) for $u(w, t)$ then takes the form

$$\frac{\partial u}{\partial t}=i\Bigg[\int_{C_\xi}\int_{C_\eta}\int_{C_\zeta}\frac{2X\overline{Y}Zd\zeta d\eta d\xi}{(w-x)^2(x-\overline{y})(z-\overline{y})}+\int_{C_\xi}\int_{C_\eta}\int_{C_\zeta}\frac{4X\overline{Y}Zd\zeta d\eta d\xi}{(w-x)(z-x)(x-\overline{y})(z-\overline{y})}+$$
$$+\int_{C_\eta}\int_{C_\zeta}\frac{2X\overline{Y}d\xi d\eta}{(x-\overline{y})(x-w)^2}-\int_{C_\zeta}\frac{2Xd\xi}{(w-x)^2}\Bigg] \qquad \begin{pmatrix} Z=X(\zeta,t) \\ z=x(\zeta,t)\end{pmatrix} \tag{17}$$

On the other hand, differentiating (11) with respect to t, we find that

$$\frac{\partial u}{\partial t}=\int_{C_\xi}\frac{1}{w-x}\frac{\partial X}{o t}d\xi+\int_{C_\xi}\frac{X}{(w-x)^2}\frac{\partial X}{\partial t}d\xi \tag{18}$$

Equating the expressions so obtained for $\partial u/\partial t$ we find that equation (10) for $u(w, t)$ is satisfied if x and X are the solution of the following system of equations:

$$\frac{\partial x}{\partial t} = 2i \int\limits_{C_\eta} \int\limits_{C_\xi} \frac{\overline{Y} Z d\zeta d\eta}{(x-\overline{y})(z-\overline{y})} + 2i \int\limits_{C_\eta} \frac{\overline{Y} d\eta}{x-\overline{y}} - 2i, \quad \frac{dX}{\partial t} = 4iX \int\limits_{C_\eta} \int\limits_{C_\zeta} \frac{\overline{Y} Z d\zeta d\eta}{(z-x)(x-\overline{y})(z-\overline{y})} \tag{19}$$

The integral with respect to ζ in the expression for $X(\xi, t)$ is interpreted in the sense of a principal value.

Theorem 1. Suppose that $\delta > 0$ and $X_0(\xi)$ is a function, defined on C_ξ and satisfying on C_ξ the conditions

$$|X_0(\xi)| \leqslant H_0, \qquad H_0 \leqslant H, \qquad |X_0(\xi_1) - X_0(\xi_2)| \leqslant H_0 |\xi_1 - \xi_2|^\mu \tag{20}$$

and for sufficiently large values of $|\xi|$ the condition

$$|X_0(\xi)| \leqslant \frac{H_0}{|\xi|^\mu} \tag{21}$$

Then, for sufficiently small t_0, the system (19) has in D_{t_0} a unique regular* solution, satisfying the initial conditions

$$x(\xi, 0) = \xi, \qquad X(\xi, 0) = X_0(\xi)$$

The fundamental difficulty in the proof of the existence of a solution of the system (19) arises from consideration of the second equation, the integrand of which has a singularity at $z = x$. In the proof, equation (19) is replaced by a system of the form

$$\frac{\partial x}{\partial t} = \int\limits_{C_\eta} \int\limits_{C_\zeta} f(x, \overline{y}, z, \overline{Y}, Z)\, d\zeta d\mu; \qquad \frac{\partial X}{\partial t} \int\limits_{C_\eta} \int\limits_{C_\zeta} \frac{\varphi(x, \overline{y}, z, \overline{Y}, Z)\, d\zeta d\eta}{z - x} \tag{22}$$

where f and ϕ are functions which are holomorphic in the region $\delta_1 \leqslant \operatorname{Im} x(\zeta, t) < \delta_2$, $0 < \delta_1 < \delta < \delta_2$, $\delta_1 < \operatorname{Im} y(\eta, t) < \delta_2$, $\delta_1 \leqslant \operatorname{Im} z(\zeta, t) < \delta_2$, $|Y| \leqslant H$, $|Z| < H$, and, for sufficiently large $|\eta|$, $|\zeta|$,

$$|Y| \leqslant \frac{H}{|\eta|^\mu}, \qquad |Z| \leqslant \frac{H}{|\zeta|^\mu}$$

Conditions, which are sufficient for the existence of a solution of the system (22), are evidently sufficient also for the existence of a solution of system (19). Moreover, since the integration with respect to η in (22) cannot worsen the convergence of the successive approximations which are employed in the proof, then for the sake of brevity the theorem of existence and uniqueness is proved for the following system of equations:

---

* A solution of the system is regular if the functions x, X satisfy the system (19) and possess the properties (13)-(15).

$$\frac{\partial x}{\partial t} = \int\limits_{C_\zeta} f(x,\ z,\ Z)\, d\zeta. \qquad \frac{\partial X}{\partial t} = \int\limits_{C_\zeta} \frac{\varphi(x,\ z,\ Z)\, d\zeta}{z - x} \tag{23}$$

where f and ϕ are functions, holomorphic in the region

$$\delta_1 \leqslant \operatorname{Im} x(\zeta,\ t) \leqslant \delta_2, \qquad \delta_1 \leqslant \operatorname{Im} z(\zeta,\ t) \leqslant \delta_2, \qquad |Z| \leqslant H, \qquad |Z| \leqslant \frac{H}{|\zeta|^{\mu}}, \quad 0 < \mu < 1$$

for sufficiently large $|\zeta|$. The behavior of the functions f and ϕ for large $|\xi|$ and $|\zeta|$ is characterized by inequalities of the form

$$|f| \leqslant \frac{M}{|\xi|} \text{ for large } |\xi|, \qquad |f| \leqslant \frac{M}{|\zeta|^{\mu+1}} \text{ for large } |\zeta|$$

If, however, ξ and ζ are simultaneously sufficiently large in modulus, then a condition of the following form is satisfied:

$$|f| \leqslant \frac{M}{|\xi|\,|\zeta|^{\mu+1}}$$

The system (23) is, in general, analogous to the system of equations studied in [2]. The essential difference, which appreciably complicates the proof, is the integration along an infinite straight line in our case, leading to the postulation of supplementary conditions concerning the sufficiently rapid decrease of the respective functions as $w \to \infty$, and to the necessity of examining the fulfilment of these conditions in approximations.

Theorem 2. Let the function $z_0(w)$ be holomorphic and one-sheeted in the half-plane Im $w < 0$ and let it satisfy the conditions

$$|z_0(w) - w| \leqslant \frac{1}{|w|^{\mu}},\ |z_0'(w) - 1| \leqslant \frac{1}{|w|^{\mu}},\ u_0(w) = \frac{1}{z_0'(w)} - 1,\ \ X_0(\xi) = \frac{u_0(\xi)}{2\pi i}$$

Moreover, let $x(\xi,\ t)$, $X(\xi,\ t)$ be the solution of system (21), regular in the region D_{t_0}, and satisfying the initial conditions

$$x(\xi,0) = \xi, \qquad X(\xi,0) = X_0(\zeta)$$

Then the function $Z(w,\ t)$ defined by formula (8), is the unique solution of the boundary problem (3) which is holomorphic in D_{t_0}.

Theorem 3. Let $Z_0(w)$ be a function, holomorphic and one-sheeted in the region Im $w < 0$. Then if the function $Z(w,\ t)$, holomorphic in the region D_T, $0 < t < T$, Im $w < 0$, is the solution of the equation $Z_t + iZ_w P = 0$ which satisfies the initial condition $Z(w,\ 0) = Z_0(w)$, then it is one-sheeted in the lower half-plane Im $w < 0$.

By means of the substitution $z = z_0(z^*)$, the proof of the Theorem is reduced to the case when $z_0(w) = w$. Then $Z(w,\ t) = \omega(0,\ w,\ t)$, where $\omega(\tau,\ w,\ t)$ is the solution of the equation $d\omega/d\tau = iP$, satisfying the initial condition $\omega(t,\ w,\ t) = w$. Since $P = iZ_t/Z_w$, then by virtue of (4) we obtain

$$\frac{d\,\mathrm{Im}\,\omega}{d\tau} = \mathrm{Im}\,(iP) < 0$$

From this inequality it follows that in the range $0 < \tau < t$ the values of ω do not lie outside the region of holomorphicity of the function $P(w, t)$. Hence, according to known theorems on the theory of differential equations, the holomorphicity and one-sheetedness of the function $Z(w, t)$ follow.

BIBLIOGRAPHY

1. Galin, L.A., *PMM* Vol. 15, No. 6, 1951.

2. Kufarev, P.P. and Vinogradov, Iu.P., *PMM* Vol. 12, No. 2, 1948.

3. Polubarinova-Kochina, P.Ia., *Teoriia dvizheniia gruntovykh vod (Theory of the Motion of Ground Waters)*. Gostekhizdat, 1952.

4. Muskhelishvili, N.I., *Nekotorye zadachi teorii uprugosti (Certain Problems in the Theory of Elasticity)*. Gostekhizdat, 1949.

Translated by A.H.A.